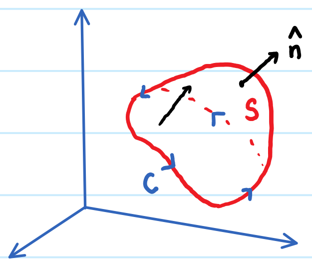
16.8 Stokes' Theorem

Higher dimensional version of Green's Theorem.

- Green's Thm relates a double integral over a plane region D to a line integral around its plane boundary curve.
- Stokes' Thm relates a surface integral over a surface S to a line integral around the boundary of S (which is a space curve).



The orientation of S induces the positive orientation of the boundary curve C. If you walk in the positive direction around C with your head pointing in the direction of $\hat{n}$, then the surface will be on your left.

Stokes' Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple closed curve C w/ positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then,

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F}\cdot d\vec{S}}$$

Since, $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\, ds$ and $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\, dS$

So Stokes' Theorem says that the line integral around the boundary curve S of the tangential component of $\vec{F}$ is equal to the surface integral over S of the normal component of the curl of $\vec{F}$.

$\partial S \equiv$ positively oriented boundary curve of oriented surface S.

Then, $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{r}$

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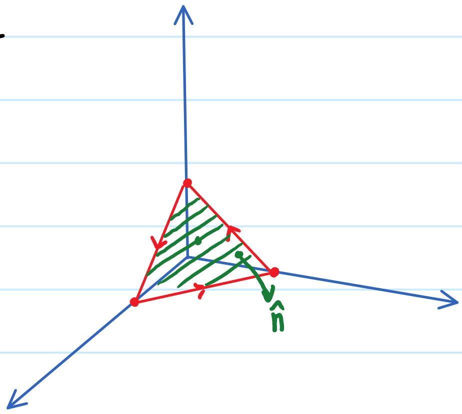
<u>Ex</u> Use Stokes theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = (x+y^2)\hat{i} + (y+z^2)\hat{j} + (z+x^2)\hat{k}$ and C is the triangle w/ vertices $(1,0,0), (0,1,0)$ and $(0,0,1)$ oriented counterclockwise when viewed from above.

<u>Soln</u>



We could compute $\int_C \vec{F}\cdot d\vec{r}$ directly, but it's easier to evaluate using Stokes' Theorem.

First, $\operatorname{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix} = -2z\,\hat{i} - 2x\,\hat{j} - 2y\,\hat{k}$

You can create many surfaces with boundary C, but the easiest one is the part of the plane enclosed by the triangle, and since we know the intercepts, the plane has equation $\frac{x}{1} + \frac{y}{1} + \frac{z}{1} = 1 \Rightarrow x+y+z = 1$.

So our surface S is given by $z = 1-x-y$, $x \geq 0$, $y \geq 0$, $z \geq 0$.
For the orientation, if we orient S upward, then C has induced positive orientation.

The projection D of S onto the xy-plane is given by $D = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1-x\}$

$g(x,y) = 1-x-y \Rightarrow \frac{\partial g}{\partial x} = -1$, $\frac{\partial g}{\partial y} = -1$.

Then, $\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F}\cdot d\vec{S} = \iint_S \left(\underbrace{-2z}_{P}\,\hat{i} \underbrace{-2x}_{Q}\,\hat{j} \underbrace{-2y}_{R}\,\hat{k}\right)\cdot d\vec{S} = \iint_D \left[-(-2z)(-1) - (-2x)(-1) - 2y\right] dA$

$\iint -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \quad dA$

$$\iint_D -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \quad dA$$

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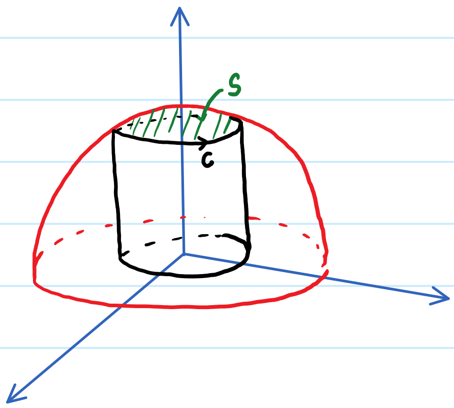
$$= \int_0^1\int_0^{1-x} -2(\underbrace{1-x-y}_{z}) - 2x - 2y \, dy\, dx = -\int_0^1 \int_0^{1-x} 2 \, dy \, dx = -\int_0^1 (2-2x)\,dx = -\left[2x - x^2\right]_0^1 = -1$$

Ex Use Stokes's Thm to compute $\iint_S \text{curl } \vec{F} \cdot d\vec{S}$ where $\vec{F}(x,y,z) = xz\,\hat{i} + yz\,\hat{j} + xy\,\hat{k}$ and S is part of the sphere $x^2+y^2+z^2=25$ that lies inside the cylinder $x^2+y^2=9$ and above the xy-plane.



To apply Stokes' Theorem, we need to find the boundary curve C.

To do that solve $x^2+y^2+z^2=25$ and $x^2+y^2=9$ and we get $z^2=16 \Rightarrow z=4$ (Since $z>0$).

Then C is the circle given by equations $x^2+y^2=9$ and $z=4$.
A vector equation for C then is $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4\rangle$, $0 \le t \le 2\pi$

- $\vec{r}\,'(t) = \langle -3\sin t, 3\cos t, 0\rangle$
- $\vec{F}(\vec{r}(t)) = \langle 12\cos t, 12\sin t, 9\cos t \sin t\rangle$

Then by Stokes' Theorem,

$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^{2\pi} -36\cos t \sin t + 36 \sin t \cos t \, dt = 0$$

Remark In above example, we only used values of $\vec{F}$ on the boundary curve C, to compute the surface integral.

This means that if we have another oriented surface w/ same boundary curve C, then we get the same value for the surface integral.

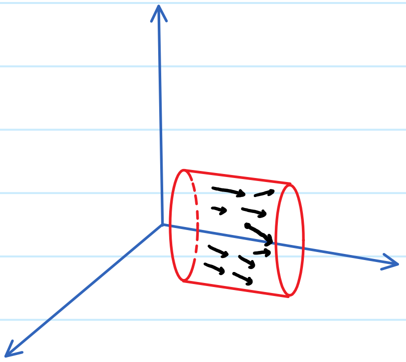
In general, if S_1, S_2 are surfaces w/ same oriented boundary curve C and both satisfy the hypothesis of Stokes' theorem. Then,

$$\iint_{S_1} \text{curl}\, \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \iint_{S_2} \text{curl}\, \vec{F} \cdot d\vec{S}$$

- Useful when it is difficult to integrate over one surface but easier on the other.

Ex

Imagine a fluid flowing steadily through surface S. Let $\vec{v}(x,y,z)$ be the velocity vector at a point (x,y,z).
Then $\vec{v}$ assigns to each point (x,y,z) in a domain E, a vector in $\mathbb{R}^3$.

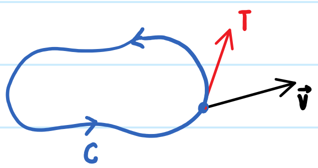


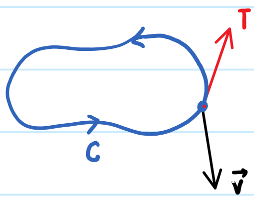
- The speed at any point is given by the length of the arrow.

Now we want to use Stokes' Theorem to shed some light on the meaning of curl vector.
Suppose C is an oriented closed curve and $\vec{v}$ represents the velocity field in fluid flow.
Consider, $\int_C \vec{v} \cdot d\vec{r} = \int_C \vec{v} \cdot \vec{T}\, ds$ where $\vec{v} \cdot \vec{T}$ is the component of $\vec{v}$ in the direction of the unit tangent vector $\vec{T}$.

This means that the closer the direction of $\vec{v}$ is to the direction of $\vec{T}$, the larger the value of $\vec{v} \cdot \vec{T}$ is.

Thus $\int_C \vec{v} \cdot d\vec{r}$ is a measure of the tendency of the fluid to move around C and is called the circulation of $\vec{v}$ around C.





$\int_C \vec{v}\cdot d\vec{r} > 0$, positive circulation $\qquad$ $\int_C \vec{v}\cdot d\vec{r} < 0$, negative circulation

- Now let $P_0(x_0, y_0, z_0)$ be a point in the fluid and S_a be a small disc w/ radius a and center P_0.

Then for any point P in S_a, $(\text{curl}\,\vec{F})(P) \approx (\text{curl}\,\vec{F})(P_0)$ (As curl $\vec{F}$ is continuous)
Then by Stokes' Theorem,

$$\int_{C_a} \vec{v}\cdot d\vec{r} = \iint_{S_a} (\text{curl}\,\vec{v})\cdot d\vec{S} = \iint_{S_a} \text{curl}\,\vec{v}\cdot\hat{n}\, dS \approx \iint_{S_a} \text{curl}\,\vec{v}(P_0)\cdot\hat{n}(P_0)\, dS = \text{curl}\,\vec{v}(P_0)\cdot\hat{n}(P_0)\,\pi a^2$$

The approximation becomes better as $a \to 0$ and we have

$$\text{curl}\,\vec{v}(P_0)\cdot\hat{n}(P_0) = \lim_{a\to 0} \frac{1}{\pi a^2}\int_{C_a} \vec{v}\cdot d\vec{r}$$

It shows that curl $\vec{v}\cdot\hat{n}$ is a measure of the rotating effect of the fluid about axis $\hat{n}$.
The curling effect is greatest about the axis parallel to curve $\vec{v}$.

<u>Thm</u> If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\vec{F} = 0$, then $\vec{F}$ is conservative.

<u>Proof</u> Recall: $\vec{F}$ is conservative if and only if $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed path C.

Given a closed path C, suppose we can find an orientable surface S whose boundary is C.*
Then by Stokes' Theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_S 0\cdot d\vec{S} = 0$$

16.9 Divergence Theorem

Recall We rewrote Green's Theorem in Vector form as :

$$\int_C \vec{F}\cdot\hat{n}\,ds = \iint_D \operatorname{div}\vec{F}(x,y)\,dA$$, where C is the positively oriented boundary curve of the plane region D.

Want to extend this to vector fields over $\mathbb{R}^3$.

Definition A region E which is simultaneously of types 1, 2 and 3 is called a simple solid region.

The boundary of E is a closed surface and the positive outward orientation.

The Divergence Theorem

Let E be a simple solid region and let S be the boundary surface of E, given with positive outward orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E.

Then, $$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV$$

- Flux of $\vec{F}$ across the boundary surface of E is equal to the triple integral of the divergence of $\vec{F}$ over E.

Ex Find the flux of $\vec{F}(x,y,z) = \langle z, y, x\rangle$ over the unit sphere $x^2+y^2+z^2 = 1$

Soln $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$.

The unit sphere S is the boundary of the unit ball B given by $x^2+y^2+z^2 \le 1$.

Then by divergence Theorem,

$$\iint_S \vec{F}\cdot d\vec{s} = \iiint_B \operatorname{div}\vec{F}\, dV = \iiint_B 1\, dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4\pi}{3}.$$

- We already knew this from an example in prvs lecture.

Ex Compute the flux of $\vec{F}(x,y,z) = \langle \cos z + xy^2, xe^{-z}, \sin y + x^2 z \rangle$ across the surface S of the solid bounded by the paraboloid $z = x^2+y^2$ and the plane $z = 4$.

Soln $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(\cos z + xy^2) + \frac{\partial}{\partial y}(xe^{-z}) + \frac{\partial}{\partial y}(\sin y + x^2 z) = x^2+y^2$

Now by divergence Thm,

$$\int_C \vec{F}\cdot d\vec{s} = \iiint_E (x^2+y^2)\, dV = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r^2\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{2} r^3(4-r^2)\, dr\, d\theta = 2\pi\left[r^4 - \frac{r^6}{6}\right]_0^2 = \frac{32\pi}{3}$$

- We can apply the divergence Theorem not only to simple solid regions but to regions that are finite union of simple solid regions.

For example, if E is a region E between closed surfaces S_1 and S_2, where S_1 lies inside S_2.

The boundary of E is $S = S_1 \cup S_2$, and it's normal $\hat{n}$ is given by $\hat{n} = -\hat{n}_1$ on S_1 and $\hat{n} = n_2$ on S_2.

Then applying divergence Thm to S, we get

$$\iiint_E \operatorname{div}\vec{F}\, dV = \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\hat{n}\, dS = \iint_{S_1} \vec{F}\cdot(-\hat{n}_1)\, dS + \iint_{S_2} \vec{F}\cdot\hat{n}_2\, dS = -\iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}$$

Ex Spse an electric charge Q is located at the origin.
Then according to Coulumb's law, the electric force $\vec{F}(\vec{x})$ exerted by this charge located at a point (x,y,z) w/ position vector $\vec{x} = \langle x,y,z \rangle$ is $\vec{F}(\vec{x}) = \frac{\varepsilon q Q}{|\vec{x}|^3}\vec{x}$, where ε is a constant.

Instead of considering force $\vec{F}$, physicists often consider force per unit charge:

$$\vec{E}(\vec{x}) = \frac{1}{q}\vec{F}(x) = \frac{\varepsilon Q}{|\vec{x}|^3}\vec{x}$$

Then $\vec{E}$ is a vector field on $\mathbb{R}^3$ called the electric field of Q.

Use the divergence theorem to show that the electric flux of $\vec{E}$ through any closed surface that encloses the origin is $\iint_{S_2} \vec{E}\cdot d\vec{S} = 4\pi\varepsilon Q$

Soln The difficulty here is that we don't have an explicit equation for S_2 because it is any closed surface enclosing the origin.

The simplest surface to work with would be a sphere, so let S_1 be a small sphere w/ radius a and center origin (contained inside S_2).

$$\operatorname{div}\vec{E} = \operatorname{div}\left[\left\langle \frac{\varepsilon Q x}{(x^2+y^2+z^2)^{3/2}}, \frac{\varepsilon Q y}{(x^2+y^2+z^2)^{3/2}}, \frac{\varepsilon Q z}{(x^2+y^2+z^2)^{3/2}} \right\rangle\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\varepsilon Q x}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{\varepsilon Q y}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{\varepsilon Q z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \varepsilon Q\left[\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{-2y^2+x^2+z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{-2z^2+x^2+y^2}{(x^2+y^2+z^2)^{5/2}}\right] = 0$$

Then, $0 = -\iint_{S_1} \vec{E}\cdot d\vec{S} + \iint_{S_2} \vec{E}\cdot d\vec{S} \Rightarrow \iint_{S_2} \vec{E}\cdot d\vec{S} = \iint_{S_1} \vec{E}\cdot d\vec{S}$

Now we can compute the surface integral over S_1.

$\hat{n}$ at $\vec{x}$ is $\dfrac{\vec{x}}{|\vec{x}|}$.

$$\vec{E}\cdot\hat{n} = \frac{\varepsilon Q}{|\vec{x}|^3}\cdot\vec{x}\cdot\left(\frac{\vec{x}}{|\vec{x}|}\right) = \frac{\varepsilon Q}{|\vec{x}|^4}|\vec{x}|^2 = \frac{\varepsilon Q}{|\vec{x}|^2} = \frac{\varepsilon Q}{a^2} \quad \left(\text{Since } S_1 \text{ is given by } |\vec{x}| = a\right)$$

Then,

$$\iint_{S_2}\vec{E}\cdot d\vec{S} = \iint_{S_1}\vec{E}\cdot d\vec{S} = \iint_{S_1}\vec{E}\cdot\hat{n}\,dS = \iint_{S_1}\frac{\varepsilon Q}{a^2}\,dS = \frac{\varepsilon Q}{a^2}\iint_{S_1}dS = \frac{\varepsilon Q}{a^2}\cdot 4\pi(a^2) = 4\pi\varepsilon Q$$

Another application of Divergence Theorem occurs in fluid flow. Let $\vec{v}(x,y,z)$ be the velocity field of a fluid w/ constant density ρ.

Then, $\vec{F} = \rho\vec{v}$ is the rate of flow (mass per unit time) per unit area.

If $P(x_0,y_0,z_0)$ is a point in the fluid and B_a be a small ball w/ center P_0 and very small radius a.

Then $\operatorname{div}\vec{F}(P) \approx \operatorname{div}\vec{F}(P_0)$ for all points in B_a since $\operatorname{div}\vec{F}$ is continuous.

Then we can approximate the flux over the boundary sphere S_a as follows:

$$\iint_{S_a}\vec{F}\cdot d\vec{S} = \iiint_{B_a}\operatorname{div}\vec{F}\,dV \approx \iiint_{B_a}\operatorname{div}\vec{F}(P_0)\,dV = \operatorname{div}\vec{F}(P_0)\iiint_{B_a}1\,dV = \operatorname{div}\vec{F}(P_0).V(B_a)$$

The approximations becomes better as $a\to 0$ and suggests that:

$$\operatorname{div}\vec{F}(P_0) = \lim_{a\to 0}\frac{1}{V(B_a)}\iint_{S_a}\vec{F}\cdot d\vec{S}$$

- It says that div $\vec{F}(P_0)$ is the net rate of outward flux per unit volume at P_0. (Hence the name divergence).

If div $\vec{F}(P_0) > 0$, the net flow is outward near P and P is called a <u>source</u>.
If div $\vec{F}(P_0) < 0$, the net flow is inward near P and P is called a <u>sink</u>.